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LXXIX. *Application of the Operational or Symbolic Calculus to the Theory of Prime Numbers.*

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§ 1. IN a series of papers [1] dating back to 1929 the operational calculus, as originally developed by O. Heaviside, was extended in different directions. Whereas Heaviside applied its methods mainly to problems connected with electrical circuits with constant parameters (independent of the time) which were initially at rest, the method could be extended to arbitrary initial conditions. But also differential equations with variable coefficients could be attacked with this method and the properties of the solutions studied. Many new relations between different functions such as Bessel functions, spherical harmonics, Laguerre's and Hermite's polynomials, and others, could thus be obtained in an extremely simple way. Also the method could be extended to a "simultaneous operational calculus" where two or more variables are treated operationally. Again, following Heaviside, the method appeared a powerful weapon for evaluating integrals.

* Communicated by the Author.

It is well known that in essence Heaviside's operational calculus and the extensions obtained later on are based on the Laplace transform. This point was stressed by J. R. Carson[2] and on several occasions by ourselves[1]. But it is, perhaps, not so well known that O. Heaviside himself plainly and clearly points this out in his book 'Electromagnetic Theory,' iii. p. 236 (London, Benn Brothers, 1922).

The great heuristic value, however, of the symbolic calculus does not seem to be generally realized yet, although a set of simple rules have been developed which make the use of it a relatively simple matter.

It is the purpose of the present note to apply the symbolic calculus this time to some problems connected with the theory of prime numbers; the ease and simplicity with which the results are obtained may, we hope, again show the great heuristic power of it. But the use of the symbolic calculus in this field shows another great advantage over the usual methods in so far that the developments can most easily be depicted *graphically*, leading to a set of graphs which, we think, may considerably clarify the properties of many discontinuous functions belonging to the theory of numbers. This possibility is brought about by the fact that the operational "image" of a discontinuous "original" is a continuous function. As, moreover, the "original" fully determines the "image" and *vice versa*[3], we can deduce the properties of the discontinuous "originals" from those of their continuous "images." The great gain thus obtained will be obvious. (A further general extension of the symbolic calculus is in course of publication.)

§ 2. If we have given a function $h(x)$ which we shall call the "*original*" we obtain its "*image*" $f(p)$ with the aid of the integral

$$f(p)=p\int_0^\infty e^{-px}h(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which relation we shall denote by

$$f(p)\doteqdot h(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

assuming at the same time, as has been usual so far in the symbolic calculus, all our originals to be zero for

$x<0$. Thus, quoting some former results[1], it is easily shown with the aid of (1) that

$$\frac{p}{p-\alpha} \doteqdot e^{\alpha x}, \quad \ldots \quad (3\,a)$$

$$\frac{1}{p} \doteqdot x, \quad \ldots \quad (3\,b)$$

$$\frac{1}{p-1} \doteqdot e^{x}-1, \quad \ldots \quad (3\,c)$$

$$\mathrm{A} \doteqdot \mathrm{A}, \quad (\mathrm{A}=\text{arbitrary constant}) \quad \ldots \quad (3\,d)$$

$$\log\frac{1}{p} \doteqdot \log x+\mathrm{C}, \quad (\mathrm{C}=\text{Euler's constant}) \quad . \quad (3\,e)$$

$$-\log(p-1) \doteqdot \mathrm{E}i(x)=\int_{\infty}^{-x}\frac{e^{-s}}{s}\,ds, \quad \ldots \quad (3\,f)$$

to which we now add

$$\log(\alpha p+\beta) \doteqdot \log\beta+\int_{\frac{\beta}{\alpha}x}^{\infty}\frac{e^{-s}}{s}\,ds=\log\beta-\mathrm{E}i\left(-\frac{\beta}{\alpha}x\right), \quad (4)$$

which latter relation follows also at once from (1).

A very important function, of which we shall make considerable use further on, is the *unit function*, which we denote by $\mathrm{U}(x-a)$ and which is zero for $x<a$, where a is a positive constant, and which equals unity for $x>a$.

Its image can, for $\mathrm{Re}\,p>0$, at once be obtained from (1) as follows:

$$p\int_{0}^{\infty} e^{-px}\mathrm{U}(x-a)\,dx=p\int_{a}^{\infty} e^{-px}.1.dx=e^{-ap},$$

or, for short:

$$e^{-ap} \doteqdot \mathrm{U}(x-a). \quad \ldots \quad (5)$$

The data from the symbolic calculus here given suffice to derive not only the " Prime number theorem," but also several other relations of the theory of numbers.

§3. Let us first consider the step function $[x]$, where, as usual, the brackets mean the greatest integer contained in the real variable x. This function is depicted in fig. 1. As indicated by the shading it can be considered as the superposition of the unit function $\mathrm{U}(x-1)$, jumping at

$x=1$, the unit function $U(x-2)$, jumping at $x=2$, etc. Thus, according to (5), we have

$$e^{-p}+e^{-2p}+e^{-3p}+\ \ldots\ \doteqdot [x],$$

or

$$\frac{1}{e^p-1} \doteqdot [x], \quad \text{Re}\, p>0. \quad . \quad . \quad . \quad . \quad (6)$$

Fig. 1.

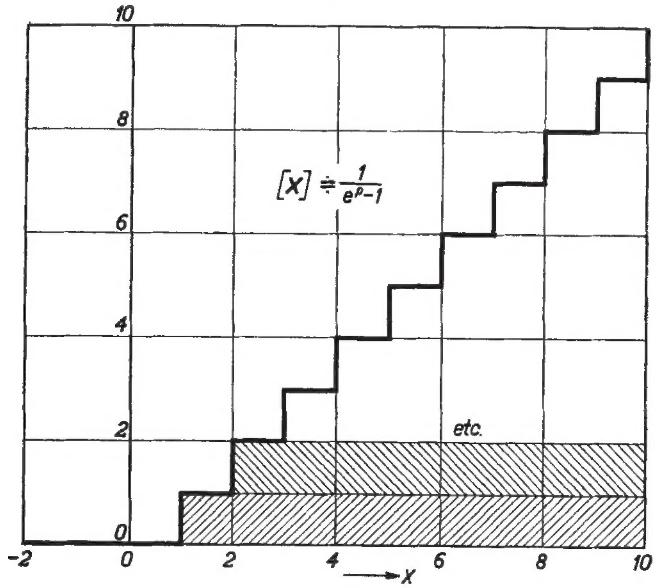
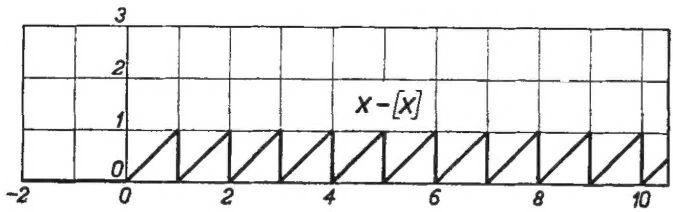


Fig. 2.



It is, further, easy to find the image of the saw-tooth function of fig. 2 given by $x-[x]$, for, with the aid of (3 *b*), we have at once

$$\frac{1}{p}-\frac{1}{e^p-1} \doteqdot x-[x], \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or, if we subtract $\frac{1}{2}$ from both sides (making the average zero),

$$x-[x]-\tfrac{1}{2} \doteqdot \frac{1}{p}-\frac{1}{e^p-1}-\tfrac{1}{2}=\frac{d}{dp}\left\{\log\left(\frac{\frac{1}{2}p}{\sinh\frac{1}{2}p}\right)\right\}, \quad (8)$$

which function of p plays an important rôle in the higher theory of the Γ functions. It is here derived, however, in an extremely simple and natural way as the image of the saw-tooth function of fig. 2.

§ 4. Consider next *the zeta function of Riemann*, $\zeta(p)$, as a function of p, and let it be the image of an original $h_1(x)$ to be obtained.

Now $\zeta(p)$ is defined for Re $p>1$ by the series

$$\zeta(p)=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots, \quad . \quad . \quad . \quad (9)$$

which we can write as

$$\zeta(p)=e^{-p\log 1}+e^{-p\log 2}+e^{-p\log 3}+\ldots\ldots \quad . \quad (9\,a)$$

But according to (5) $e^{-p\log n}$ is the image of $\mathrm{U}(x-\log n)$, *i.e.*, the function which jumps from zero to the value unity at $x=\log n$. Thus the original of $\zeta(p)$, see fig. 3, is the function which jumps a unit step at $x=\log 1$, $x=\log 2$, $x=\log 3$, etc., and therefore we can write down at once the very fundamental relation

$$\boxed{\zeta(p)\doteqdot[e^x]} \quad . \quad \text{Re}\, p>1. \quad . \quad . \quad . \quad (10)$$

It is of interest to compare this symbolic relation with the one obtained above :

$$\boxed{\frac{1}{e^p-1}\doteqdot[x]} \quad , \quad \text{Re}\, p>0 \quad . \quad . \quad . \quad (6)$$

showing that $\zeta(p)$ plays the same rôle with respect to e^x as the function $\dfrac{1}{e^p-1}$ plays with respect to x.

The fact that (10), which is here derived for the first time, could be obtained (like (6)) with the help of the symbolic calculus without practically any additional calculations again shows the great simplicity of the methods

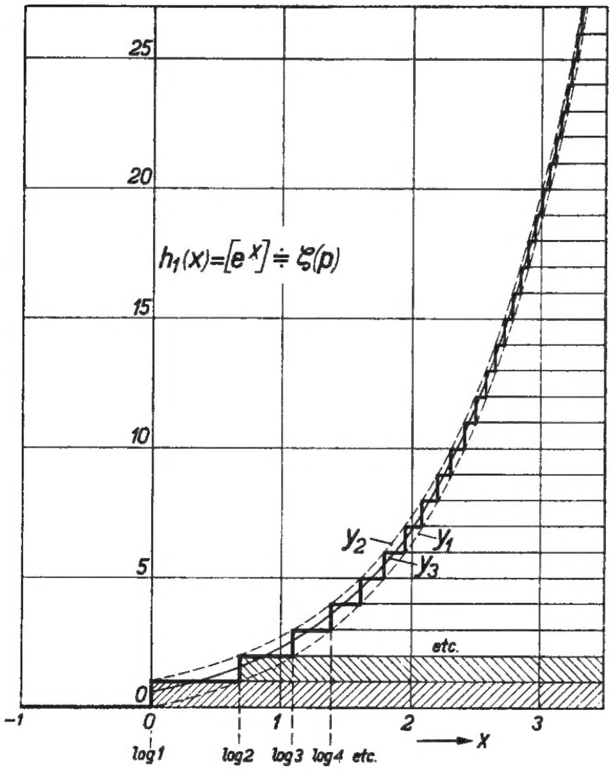
employed. From (10) with the aid of (1) we can now write at once the integral

$$\zeta(p)=p\int_0^\infty e^{-px}[e^x]\,dx,$$

or, replacing e^x by s,

$$\zeta(p)=p\int_1^\infty \frac{[s]\,ds}{s^{p+1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Fig. 3.



defining, like the basic series expansion (9), $\zeta(p)$ for Re $p>1$.

But again, as in (7), we can from (10) derive a saw-tooth function by subtracting from (10) the exponential (see 3 *a*)

$$\frac{p}{p-1} \doteqdot e^x,$$

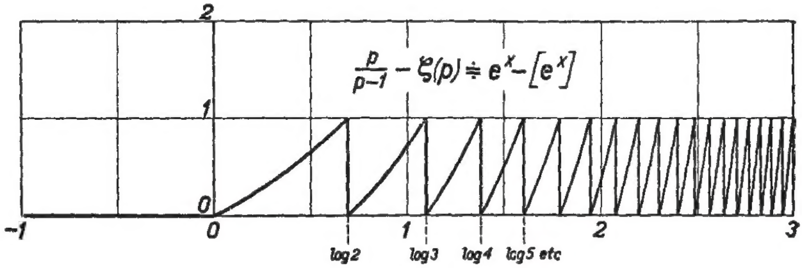
and thus obtain, see fig. 4:

$$\frac{p}{p-1}-\zeta(p) \doteqdot e^x-[e^x], \quad \operatorname{Re} p>1, \quad . \quad . \quad . \quad (12)$$

which is the analogue of (7). Now it is of interest to note that the natural construction of the saw-tooth function $e^x-[e^x]$ in (12), which oscillates between zero and unity, just annuls the one and only pole of $\zeta(p)$ at $p=1$, because it is known that the left-hand member of (12) has no poles[4]. Again, from (12) with the aid of (1) we can now write at once the integral

$$\frac{p}{p-1}-\zeta(p)=p\int_0^\infty e^{-px}\{e^x-[e^x]\}\,dx, \quad \operatorname{Re} p>1,$$

Fig. 4.



or, replacing e^x by s,

$$\frac{p}{p-1}-\zeta(p)=p\int_1 \frac{s-[s]}{s^{p+1}}\,ds, \quad . \quad . \quad . \quad (12\,a)$$

which is already valid for $0<\operatorname{Re} p$, because $(12\,a)$ represents the analytical continuation of the left-hand function which now has no poles.

Thus $(12\,a)$ extends (11) in a welcome way in so far as it converges already for $\operatorname{Re} p>0$. Hence $(12\,a)$ provides us with an integral along the real axis which defines $\zeta(p)$ also in the critical strip $0<\operatorname{Re} p<1$, in which all the much discussed zeros of the function are situated.

It is also an easy matter to construct the original of

$$\left(1-\frac{1}{2^{p-1}}\right)\cdot\zeta(p),$$

because it is equal to

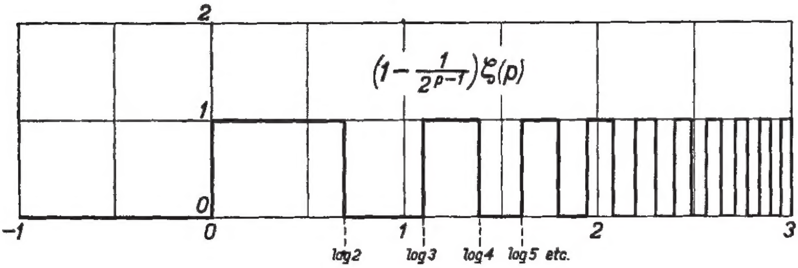
$$\sum_{n=1,2,3\,\ldots}(-1)^{n+1}\frac{1}{n^p}=\sum_n(-1)^{n+1}e^{-p\log n}, \quad . \quad . \quad . \quad (13)$$

and therefore represents the discontinuous square-topped function of fig. 5, which jumps up unity at $x = \log(2n+1)$ and down unity at $x = \log(2n)$, and thus oscillates again between 0 and $+1$. Again the image in (13), representing an original which thus remains finite everywhere, remains itself finite at $p=1$.

§ 5. In order now to derive in a most simple way the "Prime number theory" and some other relations of the theory of numbers we note, fig. 3, that

$$h_1(x) = [e^x],$$

Fig. 5.



the original of $\zeta(p)$, is everywhere contained between the two limits y_1 and y_2, where

$$y_1(x) = e^x - 1,$$
$$y_2(x) = e^x,$$

or operationally

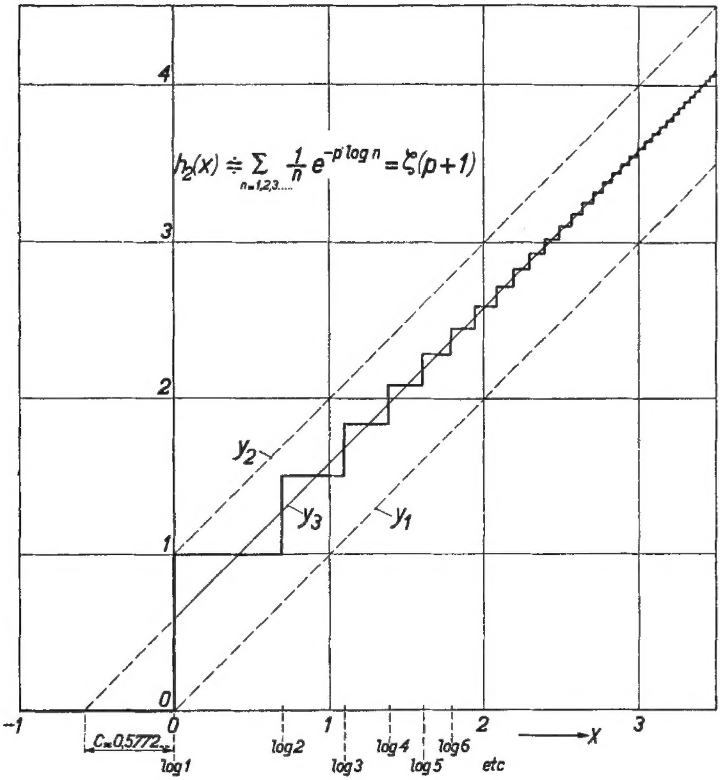
$$\left.\begin{aligned} y_1(x) &\doteqdot \frac{p}{p-1} - 1 = \frac{1}{p-1}, \\ y_2(x) &\doteqdot \frac{p}{p-1}, \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

which two functions of p we can consider as a first rough approximation to $\zeta(p)$, but where full account is taken of its one pole at $p=1$. The first expression, $\frac{1}{p-1}$, is also obtained when in (9) we replace the summation by an integration as

$$\int_1^\infty \frac{dn}{n^p} = \frac{1}{p-1}.$$

An even better approximation, $y_3(x)$, lying between $y_1(x)$ and $y_2(x)$, is at once evident from a consideration of the original $h_2(x)$ of $\zeta(p+1)$. For, see fig. 6, $\zeta(p+1)$, according

Fig. 6.



to (9), is given by

$$\zeta(p+1) = \sum_{1}^{\infty} \frac{1}{n^{p+1}} = 1e^{-p \log 1} + \tfrac{1}{2}e^{-p \log 2} + \tfrac{1}{3}e^{-p \log 3} + \ldots, \quad . \quad . \quad . \quad (15)$$

the original $h_2(x)$ of which is represented by the broken line of that figure, because, according to the rules of the symbolic calculus expounded above, the original of (15) is the function which jumps unity at $x = \log 1$, one-half at $x = \log 2$, one-third at $x = \log 3$, etc. The ordinate of this function at the jump at $x = \log n$ is therefore

$1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\frac{1}{n}$. It further tends towards a straight line. Now the definition of the Euler constant $C=0{\cdot}5772$ is

$$\lim_{n\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\frac{1}{n}-\log n\right\}=C,$$

and hence the function $h_2(x)$ of fig. 6 tends towards $x+C$. The two corresponding functions y_1 and y_2 belonging to fig. 6 derived from (14) are operationally

$$\frac{1}{(p+1)-1}=\frac{1}{p} \quad\text{and}\quad \frac{p+1}{(p+1)-1}=\frac{1}{p}+1,$$

the originals of which, as shown in fig. 6, are

$$y_1=x\doteqdot\frac{1}{p},$$

$$y_2=x+1\doteqdot\frac{1}{p}+1.$$

The better approximation is therefore $y_3=x+C$, or, in the p field,

$$y_3=x+C\doteqdot\frac{1}{p}+C.$$

Thus the better approximation in fig. 3 becomes

$$y_3\doteqdot\frac{1}{p-1}+C,$$

or

$$y_3=e^x-1+C,$$

lying, as shown in fig. 3, between y_1 and y_2.

§ 6. We have thus found two rough approximations for $\zeta(p)$, viz.,

$$y_1\doteqdot\frac{1}{p-1} \quad\text{and}\quad y_2\doteqdot\frac{p}{p-1}, \quad . \quad . \quad . \quad (14)$$

and a better one,

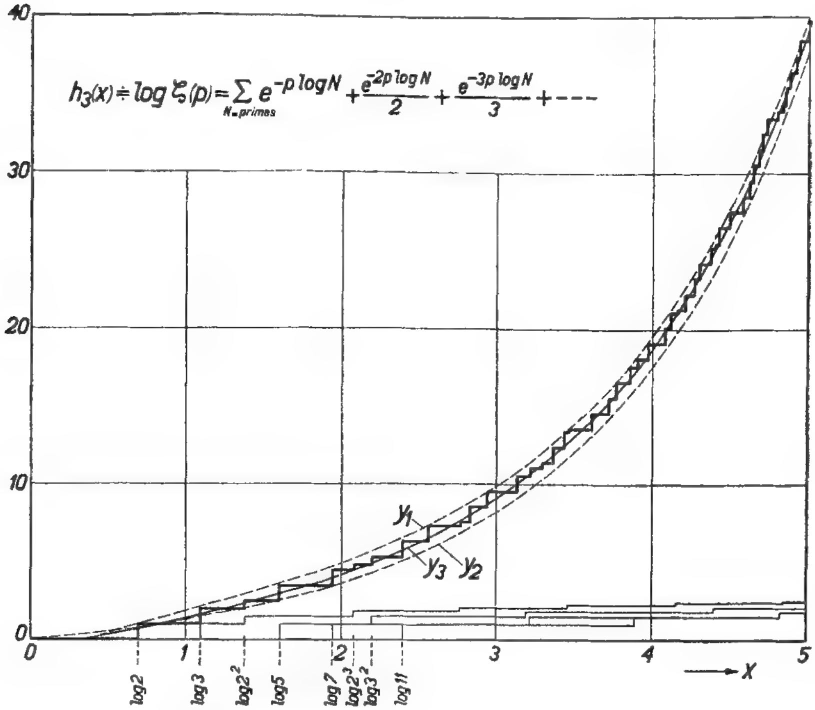
$$y_3\doteqdot\frac{1}{p-1}+C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The last expression, moreover, confirms the known result [5]:

$$\lim_{p\to1}\left(\zeta(p)-\frac{1}{p-1}\right)=C,$$

so that the approximations used here take full account of the only pole of $\zeta(p)$. It is of great interest, as we shall see further on, that these rough approximations suffice already for the derivation of several asymptotic theorems of the prime numbers, whereas the irregularities in the functions $h_1(x)$ and $h_2(x)$ of fig. 3 and fig. 4 and of the later

Fig. 7.



figures must therefore be produced by the more detailed properties, such as the zeros, of $\zeta(p)$.

§ 7. We are now ready to derive the prime number theory. We therefore consider $h_3(x)$, the original of $\log \zeta(p)$, for which according to (14) we have the rough approximations

$$y_1 \doteq \log\left(\frac{1}{p-1}\right) \quad \text{and} \quad y_2 \doteq \log\left(\frac{p}{p-1}\right) \quad . \quad (17)$$

and the better one

$$y_3 \doteqdot \log\left(\frac{1}{p-1}+\mathrm{C}\right). \quad . \quad . \quad . \quad . \quad (18)$$

Now the well-known Euler expression relating $\zeta(p)$ to the *ordinary numbers* n and the *primes* N is

$$\zeta(p) = \sum_{n=1,2,3\ldots} \frac{1}{n^p} = \prod_{\mathrm{N}=2,3,5} \left(\frac{1}{1-\frac{1}{\mathrm{N}^p}}\right), \quad . \quad (19)$$

or, taking logarithms,

$$\log \zeta(p) = \sum_{\mathrm{N}=\text{primes}} e^{-p\log \mathrm{N}} + \tfrac{1}{2}e^{-2p\log \mathrm{N}} + \tfrac{1}{3}e^{-3p\log \mathrm{N}} + \ldots . \quad . \quad . \quad . \quad (20)$$

The original $h_3(x)$ of the second member of (20) is again easily constructed (see fig. 7); for, taking the first prime, viz. 2, first, it means, as depicted at the bottom of the figure, the function which jumps unity at $x=\log 2$, one-half at $x=\log 2^2$, one-third at $x=\log 3^3$, etc. This contributing part due to the first prime 2 is represented by the top thin curve at the bottom of fig. 7. This accounts for the first prime, viz. 2. We can again construct a similar function for the second prime, viz. 3, for the third, viz. 5, etc., and add all the results. Thus the thick curve of fig. 7 is obtained. Now, according to (17) and (18) we have for $\log \zeta(p)$ the approximations

$$\left.\begin{aligned} y_1(x) &\doteqdot \log\left(\frac{1}{p-1}\right), \\ y_2(x) &\doteqdot \log\left(\frac{p}{p-1}\right), \\ y_3(x) &\doteqdot \log\left(\frac{1}{p-1}+\mathrm{C}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (21)$$

which, with the aid of (3), are seen to be equivalent to

$$\left.\begin{aligned} &y_1(x) = \mathrm{E}i(x), \\ &y_2(x) = \mathrm{E}i(x) - \log x - \mathrm{C}, \\ &y_3(x) = \log(1-\mathrm{C}) - \mathrm{E}i\left\{-x\left(\frac{1}{\mathrm{C}}-1\right)\right\} + \mathrm{E}i(x). \end{aligned}\right\}. \quad (22)$$

These functions are also depicted in fig. 7, which shows the very close approximation to the actual discontinuous

function, even for small values of x. Moreover, for $x \gg 1$ all three functions of (22) tend towards $\mathrm{E}i(x)$. But according to the construction of the discontinuous curve of fig. 7 the ordinate at $x_0 = \log \mathrm{N}_0$ (if we neglect for each prime all the jumps following the first one, which we shall call the subsidiary jumps) is equal to $\pi(\mathrm{N}_0)$, the number of primes smaller than N_0, because at each prime the function makes a unit jump. Hence for big values of x_0 we have

$$\pi(\mathrm{N}_0) \approx \mathrm{E}i(x_0) = \mathrm{E}i(\log \mathrm{N}_0),$$

and, as

$$\mathrm{E}i(x) = \int_{\infty}^{-x} \frac{e^{-s}}{s}\, ds = li(e^x),$$

where

$$li(x) = \int_0^x \frac{ds}{\log s},$$

we obtain the asymptotic *prime number theorem* :

$$\pi(\mathrm{N}) \approx li(\mathrm{N}), \qquad \ldots \ldots \quad (23)$$

or, in words : *the number of primes under* N *is asymptotically given by the logarithmic integral of* N.

It may be noted that this asymptotic result is obtained from all three approximations (21). Their difference becomes only of importance at the lower end of the curve of fig. 7, where, as could be expected, y_3 shows the optimum approximation.

§ 8. In order to derive another theorem of the theory of numbers we shall next consider the original $h_4(x)$ of $\log \zeta(p+1)$.

From (20) we first obtain

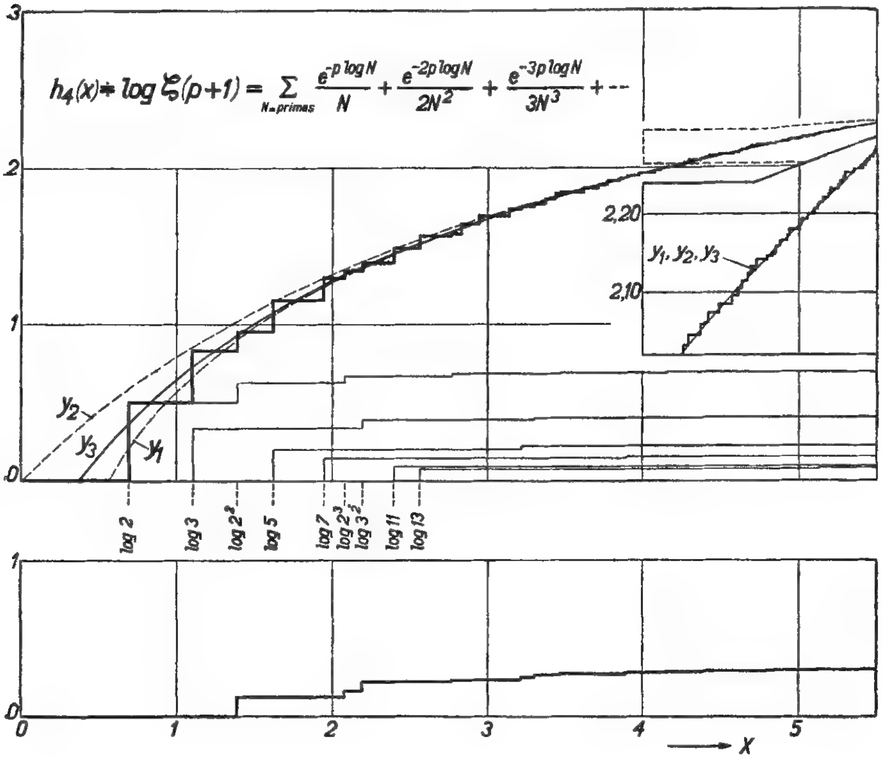
$$\log \zeta(p+1) = \sum_{\mathrm{N}=\text{primes}} \log \left(\frac{1}{1 - \frac{1}{\mathrm{N}^{p+1}}} \right)$$

$$= \sum_{\mathrm{N}} \left\{ \frac{1}{\mathrm{N}} e^{-p \log \mathrm{N}} + \frac{1}{2\mathrm{N}^2} e^{-2p \log \mathrm{N}} + \frac{1}{3\mathrm{N}^3} e^{-3p \log \mathrm{N}} + \ldots \right\}, \quad \ldots \quad (24)$$

and the contribution by each prime to the original can, as shown in fig. 8, at once again be read from (24). For, taking the prime 2 first, we obtain the upper thin curve of the upper drawing of fig. 8. It jumps unity at $x = \log 2$,

one-eighth at log 2^2, etc., and similarly for the other primes. We thus obtain the thick curve of fig. 8, where the top part of the curve is reproduced on a bigger scale in the inset.

Fig. 8.



Now from our approximations (17) and (18) to $\zeta(p)$ we have as approximations for $\log \zeta(p+1)$:

$$y_1(x) \doteqdot \log \frac{1}{p},$$

$$y_2(x) \doteqdot \log \left(\frac{1}{p}+1\right),$$

$$y_3(x) \doteqdot \log \left(\frac{1}{p}+C\right),$$

the originals of which are easily obtained as :

$$\left.\begin{aligned} y_1(x)&=\log x+\mathrm{C}, \\ y_2(x)&=\log x+\mathrm{C}-\mathrm{E}i(-x), \\ y_3(x)&=\log x+\mathrm{C}-\mathrm{E}i\left(-\frac{x}{\mathrm{C}}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

which three functions are drawn in fig. 8, which shows them to be again very good approximations.

If we neglect for the moment the subsidiary jumps, according to the construction of fig. 8, the value of the function at the abscissa $x_0=\log \mathrm{N}_0$ is $\sum\limits_{2}^{\mathrm{N}_0}\frac{1}{\mathrm{N}}$, *i. e.*, the sum of the reciprocals of all primes up to N_0. But it follows from (25) that this value must approximately be equal to $\log x_0+\mathrm{C}$ or $\log(\log \mathrm{N}_0)+\mathrm{C}$. We have in this way neglected the contribution due to the subsidiary jumps. This total subsidiary contribution is separately depicted in the bottom figure of fig. 8, and it tends towards 0·315 . . . as a numerical summation shows.

Hence we obtain the *second prime number theorem* :

$$\sum_{\mathrm{N}}^{\mathrm{N}_0}\frac{1}{\mathrm{N}}\approx\log(\log \mathrm{N}_0)+\mathrm{C}-0{\cdot}315\ldots.,$$

or

$$\mathrm{Lim}\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots\frac{1}{\mathrm{N}_0}-\log(\log \mathrm{N}_0)\right\}$$
$$\approx \mathrm{C}-0{\cdot}315\ldots, \quad . \quad . \quad (26)$$

where the first member contains the reciprocals of all prime numbers up to N_0. Expressed in words :

The sum of the reciprocals of all primes up to N_0 *minus the logarithm of the logarithm of* N_0 *asymptotically equals* C—0·315 . . ., *where* C *is Euler's constant.*

It is of interest to compare (26) with

$$\mathrm{Lim}\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.-+\frac{1}{\mathrm{N}_0}-\log \mathrm{N}_0\right\}\approx \mathrm{C},$$
$$. \quad . \quad . \quad (27)$$

where the reciprocals of all integers occur. Apart from the additional constant —0·315 . . . formula (26) is of the same form as (27), but $\log \mathrm{N}_0$ in (27) is replaced by $\log(\log \mathrm{N}_0)$ in (26).

§ 9. In order to derive still another theorem we shall consider the original of $-\frac{\zeta'(p)}{\zeta(p)}$. As approximations we obtain at once from (21)

$$y_1(x) \doteqdot \frac{1}{p-1},$$

$$y_2(x) \doteqdot \frac{1}{p-1} - \frac{1}{p},$$

$$y_3(x) \doteqdot \frac{1}{p-1} - \frac{1}{p+\frac{1-C}{C}},$$

the originals of which are

$$\left.\begin{aligned} y_1(x) &= e^x - 1, \\ y_2(x) &= e^x - 1 - x, \\ y_3(x) &= e^x - \frac{1}{1-C} + \frac{C}{1-C} e^{-\frac{1-C}{C}x} . \end{aligned}\right\} \quad . \quad . \quad (28)$$

Again, from (19) we obtain

$$-\frac{\zeta'(p)}{\zeta(p)} = \sum_{N=\text{primes}} \log N . \{e^{-p \log N} + e^{-2p \log N} + e^{-3p \log N} + \ldots .\}. \quad . \quad . \quad . \quad (29)$$

The original $h_5(x)$ of (29) can again at once be read from the form of (29), for the contribution to the total expression by each prime is a step function like fig. 1, the steps of magnitude log N occurring at $x = \log N$, $x = 2 \log N \ldots$, etc. The result is depicted in fig. 9, where also the approximations (28) are shown, and again a very good agreement is obtained. All three approximations (28) tend, for $x \gg 1$, towards e^x. Neglecting again the subsidiary steps, which for $x \gg 1$ is permitted, we thus obtain the asymptotic theorem

$$\sum_{N=2,\, 3,\, 5\, ..N_0} \log N \approx e^{x_0} = e^{\log N_0} = N_0,$$

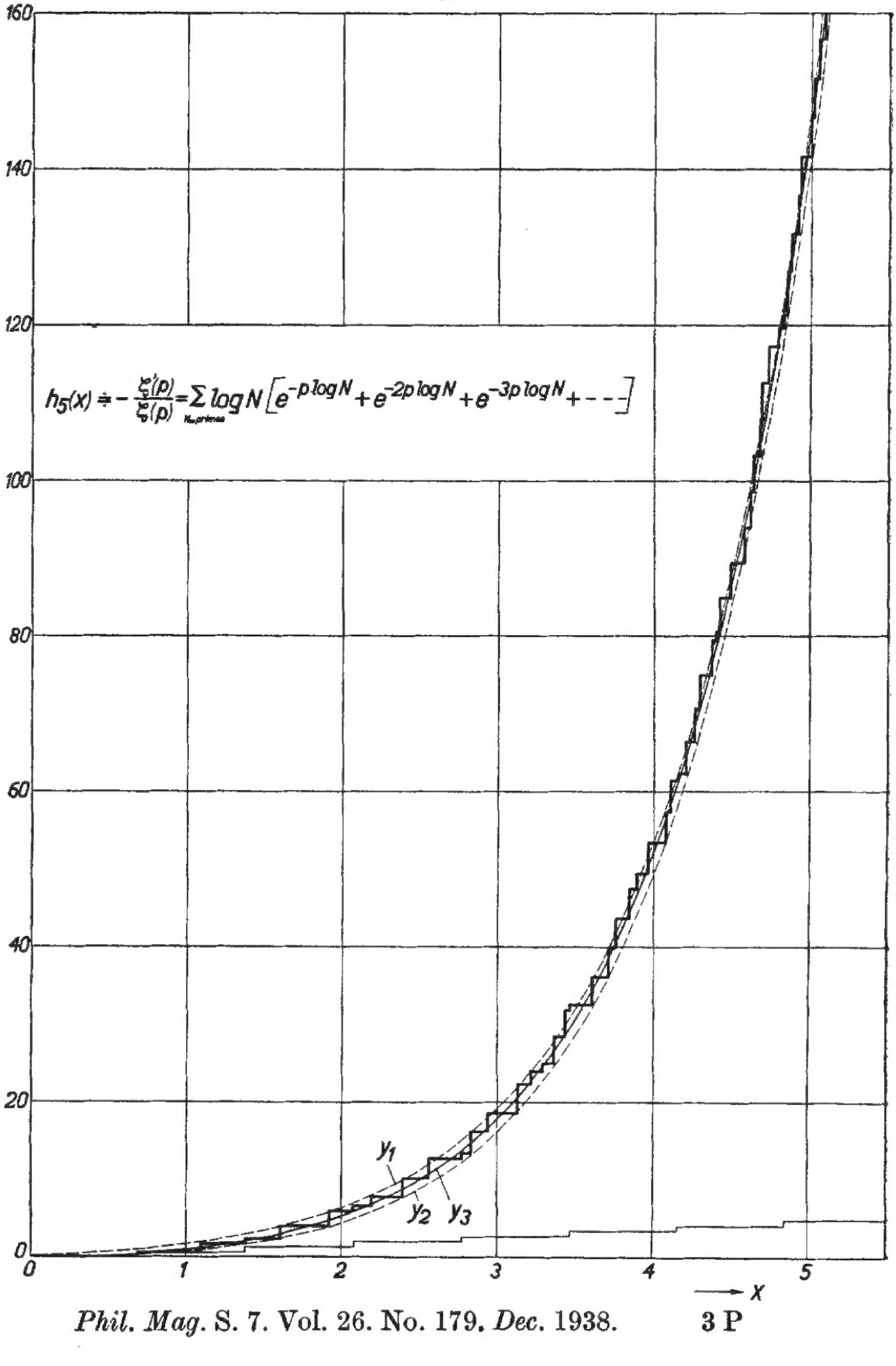
or

$$\log (2, 3, 5, 7, 11 \ldots . N_0) = N_0, \quad . \quad . \quad (30)$$

or in words :

Asymptotically any prime equals the sum of the logarithms of all smaller primes.

Fig. 9.


$h_5(x) \doteqdot -\frac{\zeta'(p)}{\zeta(p)} = \sum_{N=\text{primes}} \log N \left[e^{-p\log N} + e^{-2p\log N} + e^{-3p\log N} + \cdots\right]$
y_1
y_2
y_3
160
140
120
100
80
60
40
20
0
0
1
2
3
4
5
→ X

The simple construction of (29), which can also be written as

$$-\frac{\zeta'(p)}{\zeta(p)}=\sum_{\mathrm{N}}\log \mathrm{N}\,.\,\frac{1}{e^{p\log \mathrm{N}}-1},$$

can be shortened with the aid of (6), yielding the operational relation

$$-\frac{\zeta'(p)}{\zeta(p)}\doteqdot\sum_{\mathrm{N}}\log \mathrm{N}\,.\left[\frac{x}{\log \mathrm{N}}\right]. \quad . \quad . \quad . \quad (31)$$

§10. Finally we consider the original $h_6(x)$ of $-\dfrac{\zeta'(p+1)}{\zeta(p+1)}$. In the same way as before we obtain as approximations

$$y_1(x)\doteqdot\frac{1}{p},$$

$$y_2(x)\doteqdot\frac{1}{p}-\frac{1}{p+1},$$

$$y_3(x)\doteqdot\frac{1}{p}-\frac{\mathrm{C}}{p\mathrm{C}+1},$$

the originals of which are

$$\left.\begin{aligned} y_1(x)&=x,\\ y_2(x)&=x+e^{-x}-1,\\ y_3(x)&=x+\mathrm{C}(e^{-\frac{x}{\mathrm{C}}}-1).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (32)$$

Now we easily obtain

$$-\frac{\zeta'(p+1)}{\zeta(p+1)}$$

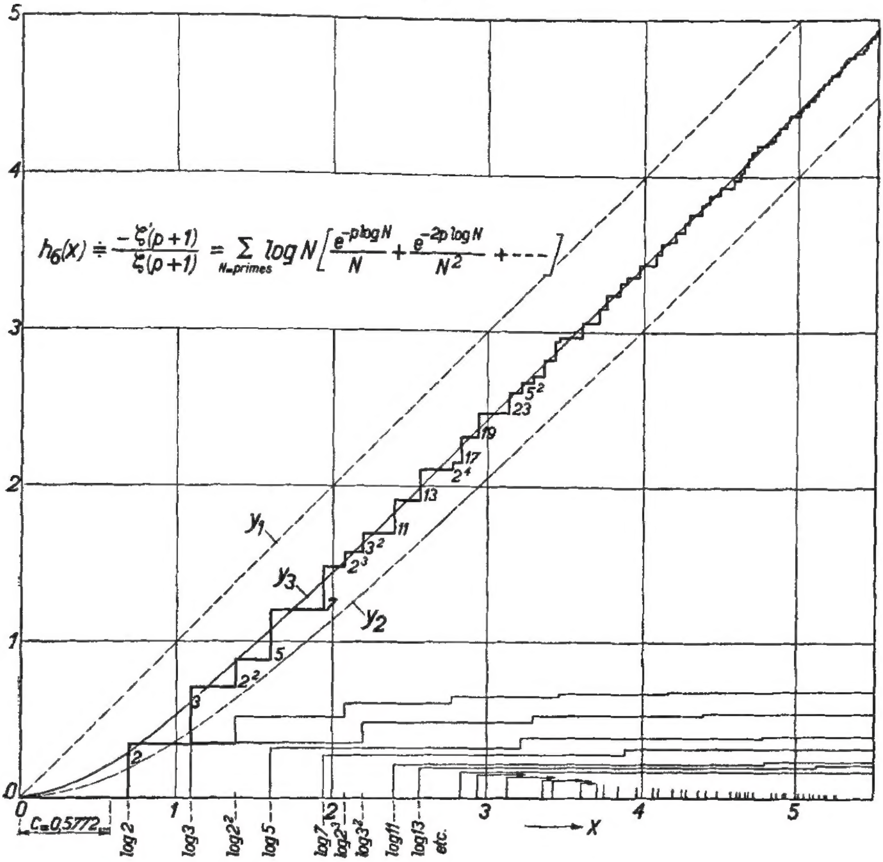
$$=\sum_{\mathrm{N}=\text{primes}}\log \mathrm{N}\,.\left\{\frac{1}{\mathrm{N}}e^{-p\log \mathrm{N}}+\frac{1}{\mathrm{N}^2}e^{-2p\log \mathrm{N}}+\ldots\right\},$$

the original $h_6(x)$ of which is shown in fig. 10, which was obtained as follows :—The contribution of the first prime, viz. 2, consists as depicted again at the bottom of the figure in a jump of magnitude $\frac{1}{2}\log 2$ at $x=\log 2$, then again a jump of $\frac{1}{2^2}\log 2$ at $x=2\log 2$, etc., and similarly for the other primes. Thus approximately a straight line is obtained from primes only, the approximations y_1, y_2 and y_3 of (32) again being shown. An even better

approximation than y_3 is given by the linear relation $y_4=x-\mathrm{C}$, as indicated in the figure. Omitting again the subsidiary jumps we thus get the asymptotic relation

$$\sum_{\mathrm{N}}^{\mathrm{N}_0}\frac{\log \mathrm{N}}{\mathrm{N}}\approx x_0=\log \mathrm{N}_0,$$

Fig. 10.



or $$2^{\frac{1}{2}}.3^{\frac{1}{3}}.5^{\frac{1}{5}}.7^{\frac{1}{7}}\ldots \mathrm{N}_0^{\frac{1}{\mathrm{N}_0}}\approx \mathrm{N}_0,$$

where, again, the product contains all the primes up to N_0. In words:

Asymptotically any prime equals the product of the N*th roots of all smaller primes* N.

In conclusion, the very good fit of all the functions considered obtained from rather rough approximations of the $\zeta(p)$ function and its derivatives, wherein mainly the pole $p=1$ is considered, and derived from $\zeta(p) \doteq [e^x_t]$, is noteworthy. Also the fact that the asymptotic theorems can thus be derived in an extremely simple way with the aid of the symbolic calculus may again be a proof of the great heuristic value of the latter.

Finally, I wish to thank Mr. C. C. J. Addink for constructing and drawing the diagrams.

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(5) E. C. Titchmarsh, *l. c.* p. 3.

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LXXX. *The Behaviour of Electrons in Iodine Vapour.*

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Introduction.

THE behaviour of electrons in chlorine and bromine has already been described in various publications †, and the present communication gives an account of an investigation of the motion of electrons in iodine, thus extending our knowledge of the electrical properties of the halogens.

* Communicated by Prof. V. A. Bailey.

† V. A. Bailey and R. H. Healey, Phil. Mag. xix. p. 725 (1935); J. E. Bailey, Makinson, and Somerville, Phil. Mag. xxiv. p. 177 (1937).